New L^6-integrability of the Hydrodynamic part and a Navier-Stokes limit

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based on a joint work with Esposito, Guo, Marra

Ki-Net Conference, April

Plan of this Presentation

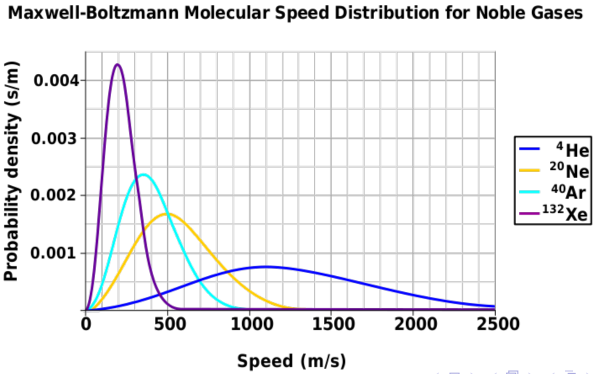
- Problems in the physical contents
  - Rarefied gas confined in a non-isothermal boundary
  - Hydrodynamic limit
- Hydrodynamic limit as a mathematical problem
  - Main theorems
  - Uniform estimate and weak limit
  - the Average Lemma and New L^6_x estimate of **Pf**
  - $L^p - L^\infty$ bootstrap

Maxwellian as an equilibrium state

The Maxwell-Boltzmann distribution:

$$PDF(|v|) = \left(\frac{m}{2\pi kT}\right)^{3/2}(4\pi|v|^2)e^{-\frac{m|v|^2}{2kT}},$$

where the particle mass m, the Boltzmann constant k, and the thermodynamic temperature T.



Non-isothermal boundary and the Non-equilibrium state

- Local Maxwellian=Equilibrium

$$M_{\rho,u,T} = \frac{\rho}{(2\pi T)^{3/2}} \exp\left(-\frac{|v-u|^2}{2T}\right)$$

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- The Boltzmann equation

$$v \cdot \nabla_x F = Q(F,F)$$

with the diffuse boundary condition

$$F|_{\gamma_-} = \sqrt{\frac{2\pi}{T_w}} M_{1,0,T_w} \int_{n\cdot u>0} F(x,u)\{n(x)\cdot u\}\mathrm{d}u$$

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- If T_w is constant then $F = M_{1,0,T_w}$.
- But the Boltzmann solution F cannot be a local Maxwellian if T_w is not constant!

Hydrodynamic limit

- Relation between F and $M_{\rho,u,T}$ where ρ, u, T solve some fluid equation such as Navier-Stokes-Fourier system.

Hydrodynamic limit

- Relation between F and $M_{\rho,u,T}$ where ρ, u, T solve some fluid equation such as Navier-Stokes-Fourier system.
- Mean free path λ is the scale of the average time that particles in the equilibrium spend traveling freely between two collisions.
- speed of sound (thermal speed) $c = \sqrt{\frac{5}{3}\frac{kT_0}{m}}$ where k the Boltzmann constant, m the molecular mass, the macroscopic reference temperature T_0.
- Dimensionless form of the Boltzmann equation

$$\mathrm{St}\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}} Q(F, F).$$

Knudsen number $\mathrm{Kn} = \frac{\lambda}{l_0}$ and Strouhal number $\mathrm{St} = \frac{l_0}{ct_0}$.

Dimensionless Form and Scalings

$$\mathrm{St}\partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\mathrm{Kn}} Q(F^\varepsilon, F^\varepsilon); \quad \mathrm{St} = \varepsilon^s, \quad \mathrm{Kn} = \varepsilon^q.$$

$$F^\varepsilon = \mu + \mathrm{Ma} f^\varepsilon \sqrt{\mu}; \quad \mathrm{Ma} := \frac{u_0}{c} = \delta(\varepsilon^m), \quad \mathrm{Re} \sim \frac{\mathrm{Ma}}{\mathrm{Kn}},$$

where $\mu = M_{1,0,1}$ and u_0 is the bulk velocity of the fluid.

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where $\mu = M_{1,0,1}$ and u_0 is the bulk velocity of the fluid.

- $q = 1,\ m = 0,\ s = 0:$ Compressible Euler ($F^\varepsilon \to F$)
- $q = 1,\ m = 1,\ s = 1:$ Incompressible Navier-Stokes-Fourier

$$\boxed{\varepsilon \partial_t F^\varepsilon + v \cdot \nabla_x F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon), \qquad F^\varepsilon = \mu + \varepsilon f^\varepsilon}$$

- $q = 1,\ m > 1,\ s = 1:$ Stokes-Fourier
- $q > 1,\ m = 1,\ s = 1:$ Incompressible Euler-Fourier

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Main theorems: Rough version

Wall temperature: $T_w = 1 + \varepsilon\theta_w$, External field: Φ

$$v \cdot \nabla_x F^\varepsilon + \varepsilon\Phi \cdot \nabla_v F^\varepsilon = \frac{1}{\varepsilon} Q(F^\varepsilon, F^\varepsilon),$$

$$F^\varepsilon|_{\gamma_-} = \sqrt{\frac{2\pi}{T_w}} M_{1,0,T_w} \int_{n\cdot u>0} F(x,u)\{n\cdot u\}\mathrm{d}u$$

Assume θ_w and Φ are small (in $H^{\frac{1}{2}}(\partial\Omega)$ and $L^2(\Omega)$).

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Assume θ_w and Φ are small (in $H^{\frac{1}{2}}(\partial\Omega)$ and $L^2(\Omega)$). Then we construct a unique solution $F^\varepsilon = \mu + \varepsilon f^\varepsilon \sqrt{\mu} \geq 0$ for $\varepsilon \ll 1$ with $\iint f^\varepsilon \sqrt{\mu} = 0$. The family $\{f^\varepsilon\}$ is weakly compact in L^2 and any limit function f equals

$$f = \{\rho + u \cdot v + \theta \frac{|v|^2 - 3}{2}\} \sqrt{\mu}$$

where ρ, u, θ satisfy the Boussinesq relation and the incompressible Navier-Stokes Fourier system with the Dirichlet boundary condition ($u = 0$, $\theta = \theta_w$ on $\partial\Omega$).

Further results

- $F^\varepsilon = \mu + \varepsilon f \sqrt{\mu} + O(\varepsilon^{3/2})$ for more regular θ_w and Φ

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- $F^\varepsilon = \mu + \varepsilon f \sqrt{\mu} + O(\varepsilon^{3/2})$ for more regular θ_w and Φ
- Moreover F^ε is asymptotically stable and the same hydrodynamic limit works for the unsteady problem
- For large initial data, if ρ, u, θ solves NSF in $t \in [0, T]$ then for $\varepsilon < \varepsilon(T)$ the hydrodynamic holds in $t \in [0, T]$.
- The Boussinesq relation

$$\nabla_x(\rho + \theta) = 0$$

- Incompressible NSF

$$\begin{aligned} u \cdot \nabla_x u + \nabla_x p = \mathfrak{v} \Delta_x u + \Phi, \quad \nabla_x \cdot u = 0 \quad &\text{in} \quad \Omega, \\ u \cdot \nabla_x \theta = \kappa \Delta_x \theta \quad &\text{in} \quad \Omega, \\ u(x) = 0, \quad \theta(x) = \theta_w(x) \quad &\text{on} \quad \partial\Omega. \end{aligned}$$

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Equation of f^ε

- Expand $Q(\mu + f^\varepsilon\sqrt{\mu}, \mu + f^\varepsilon\sqrt{\mu})$, a linear operator
$$Lf := -\frac{1}{\sqrt{\mu}}\{Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)\},$$
and the nonlinear collision operator
$$\Gamma(g, f) := \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g, \sqrt{\mu}f).$$
- $$\text{Null } L \;=\; \left\{\sqrt{\mu},\ v\sqrt{\mu},\ \frac{|v|^2-3}{2}\sqrt{\mu}\right\}.$$
- the projection onto Null L:
$$\mathbf{P}f = a(f)\sqrt{\mu} + b(f)\cdot v\sqrt{\mu} + c(f)\frac{|v|^2-3}{2}\sqrt{\mu}.$$
- semi-positivity
$$\int_{\mathbf{R}^3} fLf\,\mathrm{d}v \gtrsim \|(\mathbf{I}-\mathbf{P})f\|_{L^2_v}$$
- $\mathbf{P}L \equiv 0$, $\mathbf{P}\Gamma \equiv 0$.

Simplified problem of f^ε

- Ignore Φ and the other small terms:

$$v \cdot \nabla_x f + \varepsilon^{-1} L f = \Gamma(f, f) + (\mathbf{I} - \mathbf{P})A, \tag{1}$$

  where $A = O(1)$.

- Ignore θ_w and set the boundary condition as

$$f|_{\gamma_-} = P_\gamma f \tag{2}$$

  where

$$P_\gamma f = c_\mu \sqrt{\mu} \int_{n\cdot u>0} f \sqrt{\mu}\{n \cdot u\}\mathrm{d}u.$$

- total mass constraint $\iint f\sqrt{\mu} = 0$

Simplified problem of f^ε

- Ignore Φ and the other small terms:

$$\varepsilon^{-1} v \cdot \nabla_x f + \varepsilon^{-2} L f = \varepsilon^{-1}\Gamma(f, f) + \varepsilon^{-1}(\mathbf{I} - \mathbf{P})A,$$

  where $A = O(1)$.

- Ignore the boundary condition and set

$$\Omega = \mathbb{T}^3.$$

- total mass, momentum, energy constraints

$$0 = \iint_{\mathbb{T}^3 \times \mathbf{R}^3} f\sqrt{\mu} = \iint_{\mathbb{T}^3 \times \mathbf{R}^3} v f\sqrt{\mu} = \iint_{\mathbb{T}^3 \times \mathbf{R}^3} \frac{|v|^2 - 3}{2} f\sqrt{\mu}$$

Basic estimates

$$\boxed{\varepsilon^{-1} v \cdot \nabla_x f + \varepsilon^{-2} L f = \varepsilon^{-1} \Gamma(f,f) + \varepsilon^{-1} (\mathbf{I} - \mathbf{P}) A.}$$

- Semi-positivity $\int_{\mathbf{R}^3} f L f \,\mathrm{d}v \gtrsim \|(\mathbf{I}-\mathbf{P})f\|_{L^2_v}$
- Integration by parts, from $\mathbf{P}\Gamma = 0$

$$\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|^2_{L^2_{x,v}} \lesssim \underbrace{\varepsilon^{-1}\iint |\Gamma(f,f)(\mathbf{I}-\mathbf{P})f|}_{(\mathbf{I})} + \underbrace{\varepsilon^{-1}\iint |(\mathbf{I}-\mathbf{P})A(\mathbf{I}-\mathbf{P})f|}_{(\mathbf{II})}.$$

By Holder

$$(\mathbf{I}) \lesssim \|\Gamma(f,f)\|^2_{L^2} + o(1)\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|^2_{L^2},$$
$$(\mathbf{II}) \lesssim \|(\mathbf{I}-\mathbf{P})A\|^2_{L^2} + o(1)\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|^2_{L^2}.$$

We need higher integrability of $\mathbf{P}f$

$$\boxed{\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_{L^2}^2 \lesssim \|\Gamma(f,f)\|_{L^2}^2 + O(1)}$$

$$\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_{L^2}^2 \lesssim \||f|^2\|_{L^2}^2 + O(1)$$

- $f = \mathbf{P} + (\mathbf{I}-\mathbf{P})f$
- $\||f|^2\|_{L^2}^2 = \underbrace{\Big\||f||(\mathbf{I}-\mathbf{P})f|\Big\|_{L^2}^2}_{(\mathbf{I})} + \underbrace{\||\mathbf{P}f|^2\|_{L^2}^2}_{(\mathbf{II})}$
- $\Big\||f||(\mathbf{I}-\mathbf{P})f|\Big\|_{L^2}^2 \leq \varepsilon^2\|f\|_\infty^2\varepsilon^{-2}\|(\mathbf{I}-\mathbf{P})f\|_{L^2}^2$. If $\varepsilon\|f\|_\infty \sim o(1)$ then $(\mathbf{I})$ is fine.
- We need $\||\mathbf{P}f|^2\|_{L^2}^2 = \|\mathbf{P}f\|_{L^4}^4 \lesssim 1$!

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Average lemma: Golse-Lions-Perthame-Sentis (1988)

- Recall $\mathbf{P}f$: the L^2_v-projection onto the null space pf L
- $\mathbf{P}f \sim \int f\phi(v)\mathrm{d}v$ where ϕ is almost like C_c^∞

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- Average lemma: If
$$v \cdot \nabla_x f = g$$
then
$$\Big\| \int f\phi \mathrm{d}v \Big\|_{H_x^{1/2}} \lesssim \|f\|_{L^2} + \|g\|_{L^2}$$

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  DiPerna–Lions (Ann. Math. 1989), Golse–Saint-Raymond (Invent. Math. 2004)
- Sobolev embedding in $3D$: $H^{1/2} \subset L^3$. Not enough!

Proof of the Average Lemma

By Fourier transform in x,

$$\hat{f}(\xi, v) = \frac{\hat{g}(\xi, v)}{i(v \cdot \xi)} \tag{3}$$

With $\alpha > 0$,

$$\Big| \int_{\mathbf{R}^3} \hat{f}(\xi, v)\phi(v)\mathrm{d}v \Big| \leq \int \mathbf{1}_{|v\cdot\xi|\leq\alpha} + \int \mathbf{1}_{|v\cdot\xi|\geq\alpha}$$

If $|v \cdot \xi| \leq \alpha$ then $\left(\int \mathbf{1}_{|v\cdot\frac{\xi}{|\xi|}|\leq\frac{\alpha}{|\xi|}}\phi(v)^2\mathrm{d}v\right)^{1/2} \lesssim \alpha^{1/2}|\xi|^{-1/2}$. Then

$$\int \mathbf{1}_{|v\cdot\xi|\leq\alpha} \lesssim \alpha^{1/2}|\xi|^{-1/2}\Big(\int_v |\hat{f}|^2\Big)^{1/2}$$

For $|v \cdot \xi| \geq \alpha$, from (3), $\int \mathbf{1}_{|v\cdot\xi|\geq\alpha}$ is bounded as

$$\Big(\int_v |\hat{g}|^2\Big)^{1/2}\Big(\int_v \mathbf{1}_{|v\cdot\frac{\xi}{|\xi|}|\geq\frac{\alpha}{|\xi|}}\frac{1}{|\xi|^2}\frac{|\phi(v)|^2}{|v\cdot\frac{\xi}{|\xi|}|^2}\Big)^{1/2}$$

$$\lesssim \Big(\int_v |\hat{g}|^2\Big)^{1/2}\frac{1}{|\xi|}\Big(\int_{\frac{\alpha}{|\xi|}}^{10}\frac{dt}{t^2}\Big)^{\frac{1}{2}} \lesssim \Big(\int_v |\hat{g}|^2\Big)^{1/2}\frac{1}{(\alpha|\xi|)^{1/2}}$$

Proof of the Average Lemma

We choose

$$\alpha = \Big(\int_v |\hat{g}|^2 \Big)^{1/2} \Big/ \Big(\int_v |\hat{f}|^2 \Big)^{1/2}$$

Then

$$\Big| \int_{\mathbf{R}^3} \hat{f}(\xi, v)\phi(v)\mathrm{d}v \Big| \lesssim \frac{1}{|\xi|^{1/2}} \Big(\int_v |\hat{f}|^2 \Big)^{1/4} \Big(\int_v |\hat{g}|^2 \Big)^{1/4}$$

and

$$\Big\| |\xi|^{1/2} \int_v \hat{f}(\xi)\phi \Big\|_{L^2} \lesssim \|f\|_{L^2}^{1/2} \|g\|_{L^2}^{1/2}.$$

New L^6 estimate of $\mathbf{P}f$

If

$$v \cdot \nabla_x f = g \quad \text{in} \;\; \mathbb{T}^3 \times \mathbf{R}^3$$

and $\iint f(1, v, \frac{|v|^2-3}{2})\sqrt{\mu}\mathrm{d}v\mathrm{d}x = 0$ then

$$\|\mathbf{P}f\|_{L^6_x} \lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^6} + \|g\|_{L^2}.$$

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$$\|\mathbf{P}f\|_{L^6_x} \lesssim \|(\mathbf{I} - \mathbf{P})f\|_{L^6} + \|g\|_{L^2}.$$

- Remark: The proof is based on the test function method in Esposito-Guo-K-Marra (2013)

Proof of new L^6 bound

- Set the test function $\psi(x)\phi(v)$. Then by the integration by parts

$$-\iint_{\mathbb{T}^3\times\mathbf{R}^3} f\phi(v)v\cdot\nabla_x\psi(x) = \iint_{\mathbb{T}^3\times\mathbf{R}^3} g\psi(x)\phi(v)$$

- $f = \mathbf{P}f + (\mathbf{I}-\mathbf{P})f$ and $\mathbf{P}f = \{a + v\cdot b + c\frac{|v|^2-3}{2}\}\sqrt{\mu}$.

$$\begin{aligned} &\iint \{a + v\cdot b + c\frac{|v|^2-3}{2}\}\sqrt{\mu}\phi(v)v\cdot\nabla_x\psi(x) \\ \sim\ &\iint (\mathbf{I}-\mathbf{P})f\phi(v)v\cdot\nabla_x\psi(x) + \iint g\psi\phi \end{aligned}$$

Why L^6

- In Esposito-Guo-K-Marra (2013)

$$
\begin{aligned}
(|v|^2 - \beta_a)\{v \cdot \nabla_x\}(-\Delta_N)^{-1} a, \\
(v_i^2 - \beta_b)\sqrt{\mu}\partial_j(-\Delta_N)^{-1} b_j, \\
v_i v_j |v|^2 \sqrt{\mu}\partial_j(-\Delta_N)^{-1} b_i, \quad i \neq j, \\
(|v|^2 - \beta_c)\sqrt{\mu} v \cdot \nabla_x(-\Delta_N)^{-1} c
\end{aligned}
$$

we have

$$
\|\mathbf{P}f\|_{L^2}^2 \lesssim \iint |(\mathbf{I}-\mathbf{P})f||\mathbf{P}f| + \iint \underbrace{|g|}_{L^2} \underbrace{|\nabla_x(-\Delta_N)^{-1}\mathbf{P}f|}_{\text{gain one derivative}}
$$

- We set $\psi_a(x) = \nabla_x(-\Delta)^{-1}(a^{p-1} - \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} a^{p-1})$ with the same choice of $\phi(v)$
-

$$\int a^p + \int a \times \frac{1}{|\mathbb{T}^3|}\int_{\mathbb{T}^3} a^{p-1}$$
$$\lesssim \iint |(\mathbf{I}-\mathbf{P})f||a|^{p-1} + \iint g\nabla_x(-\Delta)^{-1}a^{p-1}$$

- $a^{p-1} \in L^{\frac{p}{p-1}}$, $\nabla_x(-\Delta)^{-1}a^{p-1} \in W^{1,\frac{p}{p-1}} \subset L^{\frac{3p}{2p-3}}$ in 3D
- We want $\frac{3p}{2p-3} = 2$ and hence $p = 6$.

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$L^p - L^\infty$ bootstrap argument

$$\boxed{v \cdot \nabla_x f + \varepsilon^{-1} L f = \Gamma(f,f)}$$

- $Lf \sim f - \int_{|u|\le N} f(u)\mathrm{d}u$
- $v \cdot \nabla_x f + \varepsilon^{-1} f = \varepsilon^{-1} \int_{|u|\le N} f(u)\mathrm{d}u + \Gamma(f,f)$
- Along the trajectory

$$\frac{d}{ds}\left(e^{-\frac{t-s}{\varepsilon}} f(x-(t-s)v, v)\right)$$
$$= e^{-\frac{t-s}{\varepsilon}} \left\{ \varepsilon^{-1} \int_{|u|\le N} f(x-(t-s)v, u) + \Gamma(f,f)(x-(t-s)v, v) \right\}$$

-

$$f(x,v) = e^{-\frac{t}{\varepsilon}} f(x - tv, v) + \int_0^t \frac{e^{-\frac{t-s}{\varepsilon}}}{\varepsilon} \int_{|u|\le N} f(x-(t-s)v, u).$$

$L^p - L^\infty$ bootstrap argument

- Iterate once again

$$f(x,v) \sim \int_0^t \mathrm{d}s \frac{e^{-\frac{t-s}{\varepsilon}}}{\varepsilon} \int_{|u|\leq N} \mathrm{d}u \int_0^s \mathrm{d}s' \frac{e^{-\frac{s-s'}{\varepsilon}}}{\varepsilon} \times \int_{|u'|\leq N} \mathrm{d}u' f(x-(t-s)v-(s-s')u, u')$$

- $u \mapsto X = x-(t-s)v-(s-s')u$ and $\mathrm{d}u \lesssim \frac{1}{|s-s'|^d}\mathrm{d}X$ in $\mathbf{R}^d$
-
-

$L^p - L^\infty$ bootstrap argument

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$$f(x,v) \sim \int_0^t \mathrm{d}s \frac{e^{-\frac{t-s}{\varepsilon}}}{\varepsilon} \int_{|u|\le N} \mathrm{d}u \int_0^{s-o(1)\varepsilon} \mathrm{d}s' \frac{e^{-\frac{s-s'}{\varepsilon}}}{\varepsilon} \times \int_{|u'|\le N} \mathrm{d}u' f(x-(t-s)v-(s-s')u, u')$$

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- $u \mapsto X = x-(t-s)v-(s-s')u$ and $\mathrm{d}u \lesssim \frac{1}{\varepsilon^d}\mathrm{d}X$ in $\mathbf{R}^d$

$L^p - L^\infty$ bootstrap argument

- Iterate once again

$$f(x,v) \sim \int_0^t \mathrm{d}s \frac{e^{-\frac{t-s}{\varepsilon}}}{\varepsilon} \int_{|u|\leq N} \mathrm{d}u \int_0^{s-o(1)\varepsilon} \mathrm{d}s' \frac{e^{-\frac{s-s'}{\varepsilon}}}{\varepsilon} \times \int_{|u'|\leq N} \mathrm{d}u' f(x-(t-s)v-(s-s')u, u')$$

- $u \mapsto X = x-(t-s)v-(s-s')u$ and $\mathrm{d}u \lesssim \frac{1}{\varepsilon^d}\mathrm{d}X$ in $\mathbf{R}^d$
- $\|f\|_\infty \lesssim \frac{1}{\varepsilon^{d/p}}\|f\|_{L^p(\mathbb{T}^d\times\mathbf{R}^d)}$

$L^p - L^\infty$ bootstrap argument

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- $\|f\|_\infty \lesssim \frac{1}{\varepsilon^{d/p}}\|f\|_{L^p(\mathbb{T}^d\times\mathbf{R}^d)}$
- Recall that $\|\mathbf{P}f\|_{L^6} + \varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2} \lesssim 1$. Therefore, for $d=3$

$$\|f\|_\infty \lesssim \frac{1}{\varepsilon^{3/2}}\|(\mathbf{I}-\mathbf{P})\|_{L^2} + \frac{1}{\varepsilon^{3/6}}\|\mathbf{P}f\|_{L^6} \lesssim \varepsilon^{-1/2}$$

Specular BC

- Shizuta-Asano (1977): specular BC in smooth convex domains without a proof
- Guo (2010): the first proof under the strong condition of the (real) analyticity of $\partial\Omega$ based on the $L^p - L^\infty$ bootstrap argument
- K-Lee (2016): complete proof

Close the estimates

- Collecting all the estimates,

$$\begin{aligned}
\varepsilon^{1/2}\|f\|_\infty &\lesssim 1 \\
\|\mathbf{P}f\|_{L^6_x} &\lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^6} + \|\Gamma(f,f)\|_{L^2} \\
\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2} &\lesssim \|\Gamma(f,f)\|_{L^2} + O(1) \\
\|\Gamma(f,f)\|_{L^2} &\lesssim \varepsilon^{1/2}\varepsilon^{1/2}\|f\|_\infty\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2} + \|\mathbf{P}f\|_{L^4}.
\end{aligned}$$

- $\|(\mathbf{I}-\mathbf{P})f\|_{L^6} \lesssim [\varepsilon^{-1}\|(\mathbf{I}-\mathbf{P})f\|_{L^2}]^{1/3}[\varepsilon^{1/2}\|f\|_\infty]^{2/3} \lesssim 1$

Thank you!